# What is the heat kernel?

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## Contents



## 1 Introduction

We begin with the definition of a linear integral operation and a notion of a kernel of an integral operator. Some illustrative examples and basic properties will be provided. Before we define the heat kernel on a compact manifold, we construct the explicit formulas for some model spaces such as Euclidian space and hyperbolic space. In the end the main idea of the heat kernel approach to the proof of the index theorem will be given.

Let $M$ be a compact manifold with a non-negative measure $\mu$.

## 2 Integral operator on a manifold

**Definition 1.** Let $k : M \times M \to \mathbb{R}$ be a continuous function. An *linear integral operator* $L_k$ with *kernel* $k$ is formally defined by

$$L_k f(x) = \int_M k(x,y) f(y) d\mu(d), \tag{2.1}$$

where $f : M \to \mathbb{R}$ is a continuous function.

*Remark.* For a compact manifold 2.1 is indeed a linear bounded operator on $C(M)$. For a non-compact manifold see [J]

An integral operator is a generalisation of an ordinary matrix multiplication. Let $A = (a_{ij})$ be $n \times n$ matrix and let $u \in \mathbb{R}^n$ be a vector. Then $Au$ is also a n-dimensional vector and

$$(Au)_i = \sum_{i=1}^{n} a_{ij} u_j, \quad i = 1, ..., n.$$

Thus, the function values $k(x, y)$ are analogous to the entries $a_{ij}$ of the matrix $A$, and the values $L_k f(x)$ are analogous to the entries $(Au)_i$.

**Example 2.1.** Consider the Volterrs operator on $L^2[0, 1]$ defined by

$$Lf(x) := \int_0^x f(y)dy \quad f \in L^2[0, 1].$$

It is an integral operator on $L^2[0, 1]$ with kernel

$$k(x, y) = \begin{cases} 1, & \text{for } x \leq y \\ 0, & \text{for } y > x \end{cases}$$

# 3 Heat kernel

## 3.1 Heat equation

**Definition 2.** Let $M$ be a compact Riemannian manifold and $\Delta$ the Laplace-Beltrami operator. The *heat equation* is the following partial differential equation

$$\frac{\partial u}{\partial t} + \Delta u = 0,$$

where $u : [0, \infty) \times M \to \mathbb{R}$ is a smooth function.

Consider a smooth function $f : M \to \mathbb{R}$ and the heat equation with the initial condition.

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta u = 0 \\ u(0, x) = f(x) \end{cases} \tag{3.1}$$

There exists a unique smooth solution for the initial value problem 3.1 which exists for all $t \geq 0$ (See [R]). The solution is $u(t, x) = e^{-t\Delta} f(x)$.

**Definition 3.** By the functional calculus there exists a continuous function $k_t(x, y : (0, \infty)) \times M \times M \to \mathbb{R}$ called *the heat kernel* such that

$$e^{-t\Delta}u(x) = \int_M k_t(x, y)u(y)d\mu(y),$$

for all smooth functions $u(x)$ and $t > 0$.

**Proposition 3.1.** *The heat kernel has the following properties:*

*(1)* $(\frac{\partial}{\partial t} + \Delta)k_t(x, y) = 0$*, i.e. for each fixed* $y \in M$ *the function* $x \mapsto k_t(x, y)$ *satisfies the heat equation;*

*(2) for each smooth function* $u(x)$*,* $\int_M k_t(x, y)u(y)d\mu(y) \to u(x)$ *uniformly at* $x$*.*

*Moreover, the heat kernel is the unique time-dependent function which is* $C^2$ *in* $x$ *and* $y$*,* $C^1$ *in* $t$ *and has properties* (1) *and* (2)*.*

**Example 3.2.** (Euclidian space)

Consider 3.1 on $\mathbb{R}^n$ with bounded $f(x)$ the solution of has the following form $u(t, x) = \int_{\mathbb{R}^n} k(t, x, y)f(y)dy$ with

$$k(t, x, y) = \frac{1}{(4\pi t)^n/2} e^{-\frac{|x-y|^2}{4t}}. \tag{3.2}$$

**Example 3.3.** (Hyperbolic space)

The heat kernel on the three-dimensional hyperbolic space $\mathbb{H}^3$

$$k(t, x, y) = \frac{1}{(4\pi t)^3/2} e^{-\frac{d(x,y)^2}{4t} - t} \frac{d(x, y)}{\sinh(d(x, y))}, \tag{3.3}$$

where $d(x, y)$ is the geodesic distance.

It is interesting to study the heat kernel from the geometric point of view because as we can already see on the examples the heat kernel is sensitive to the geometry of a manifold. However explicit formulas exist only for a few classes of manifolds with enough symmetries.

## 3.2 Asymptotic expansion

**Definition 4.** Let $f$ be a function on $\mathbb{R}_+$ with values in a Banch space $E$. A formal series

$$f(t) \sim \sum_{k=0}^{\infty} a_k(t),$$

where $a_k : \mathbb{R}_+ \to E$, is called an *asymptotic expansion* for $f$ near $t = 0$ if for each $n \in \mathbb{N}$ there exists $l_n$ such that, for all $l \geq l_n$ there is a constant $C_{l,n}$ such that

$$||f(t) - \sum_{k=0}^{l} a_k(t)|| \leq C_{l,n}|t|^n$$

for sufficiently small $t$.

**Example 3.4.** Taylor series for a function $f(t) \in C^\infty$: $f(t) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$ is an asymptotic expansion, but it is convergent only if $f$ is analytic near zero. So we see that an asymptotic expansion does not have to converge to the function itself.

**Definition 5.** An *approximate heat kernel* of order $m > 0$ is a function $\tilde{k}_t(x, y)$ which is $C^1$ in $t$ and $C^2$ in $x$ and $y$, which has the property 3.1(2) and approximately satisfies the heat equation in the following sence

$$(\frac{\partial}{\partial t} + \Delta)\tilde{k}_t(x, y) = t^m r_t(x, y),$$

where $r_t(x, y)$ is a function which depends continuously on $t$ for $t > 0$.

**Proposition 3.5.** *Let $k_t(x, y)$ be a heat kernel on $M$. For any $m > 0$ there is $\tilde{m} \geq m$ such that if $\tilde{k}_t(x, y)$ is an approximate heat kernel of order $\tilde{m}$ then*

$$k_t(x, y) - \tilde{k}_t(x, y) = t^m e_t(x, y),$$

*where $e_t(x, y)$ depends continuously on $t$ for $t/geq0$ and in $C^m$ in $x$ and $y$.*

We build the approximate heat kernel from local data.

**Theorem 3.6.** *Let $M$ be a compact manifold, $k_t$ - the heat kernel then*

*(1) there is an asymptotic expansion*

$$k_t(x, y) \sim h_t(x, y) \left(a_0(x, y) + ta_1(x, y) + t^2 a_2(x, y) + ...\right),$$

*(2) it may be differentiated,*

*(3) the values $a_i(x, x)$ along the diagonal can be computed by algebraic expressions involving the metrics and connection coefficients and their derivatives, moreover $a_0(x, x) = 1$.*

In principle it is possible to compute all the coefficients $a_i$ just by following through the proof. But in practice the details soon become exhausting.

### 3.3 The index theorem

Here is the idea of the proof of the index theorem using the heat kernel.

Let $L_k$ be an integral operator a smooth heat kernel $k_t(x, y)$. Define a trace of an operator as the sum of its eigenvalues and let $Tr_s$ be a supertrace. The following theorem can be proven.

**Theorem 3.7.** $Tr_s(L_k) = \int_M k_t(x, x) d\mu(x)$.

**Definition 6.** Let $D$ be graded operator $D_+ : C^\infty(S_+) \to C^\infty(S_-)$ and $D_- := D_+^*$. The *index* of an operator is $\mathrm{Ind}(D) := \dim\ker D_+ - \dim\ker D_-$.

Now let $\Delta^+ := D_- D_+$ and $\Delta^- := D_+ D_-$ then $\mathrm{Tr}_s e^{-tD^2} = \mathrm{Tr} e^{-t\Delta^+} - \mathrm{Tr} e^{-t\Delta^-}$.

**Proposition 3.8.** *McKean-Singer formula*

$$Ind(D) = Tr_s e^{-tD^2}$$

We see that the left hand side of the McKean-Singer formula does not depend on $t$, while the right hand side depends on $t$. Since the formula is true for any $t$ we can assume that it tends to zero. Using the asymptotic expansion of the heat kernel

$$\mathrm{Tr}_s e^{-tD^2} = \int_M k_t(x, x) d\mu(x) \sim \frac{1}{(4\pi t)^{n/2}} \left( \int \mathrm{tr}_s a_0 d\mu + \int \mathrm{tr}_s t a_1 d\mu + ... \right)$$

we obtain the index theorem

**Theorem 3.9.** *If the dimension $n$ is even then*

$$IndD = \frac{1}{(4\pi)^{n/2}} \int_M tr_s a_{n/2} d\mu$$

*otherwise $IndD = 0$.*

# References


[J] Jörgens, Konrad (1982). Linear integral operators. Surveys and Reference Works in Mathematics 7. Boston, Mass.-London: Pitman (Advanced Publishing Program).

[R] J. Roe, Elliptic operators, topology and asymptotic methods, Second edition, Chapman and Hall/CRC (1999)